On the equivalence of fractional-order Sobolev semi-norms *

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Abstract

We present various results on the equivalence and scaling properties of fractional-order Sobolev norms and semi-norms of orders between zero and one. Main results are mutual estimates of the three semi-norms of Sobolev-Slobodeckij, interpolation and quotient space types. In particular, we show that the former two are uniformly equivalent under scaling.

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1 Introduction

Sobolev norms and semi-norms play a central role in the numerical analysis of discretization methods for partial differential equations. For instance, standard finite element error analysis is essentially a combination of the Bramble-Hilbert lemma and transformation properties of Sobolev (semi-) norms. These properties are also central to the area of preconditioners for (and based on) variational methods. More precisely, arguments based on finite dimensions of local spaces are inherently connected with scaling arguments to keep dimensions bounded. Norms are usually not scalable, i.e. the corresponding equivalence numbers behave differently with respect to a scaling parameter when the domain under consideration is isotropically scaled. This can be usually fixed only when essential boundary conditions are present. An example is using the H^1-semi-norm as norm in H^1_0. More generally, semi-norms have better scaling properties: usually they can be defined so that equivalence numbers are identical under isotropic scaling of the domain.

Whereas properties of Sobolev (semi-) norms under smooth transformations or simple scalings are straightforward as long as their orders are integer, things are getting more complicated for fractional-order Sobolev norms. Such norms appear in a natural way when considering boundary integral equations of the first kind. For an overview see, e.g., [10, 8]. There are

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different ways to define fractional-order Sobolev norms and they all have advantages and disadvantages (standard references are [9, 1]). Different norm variants are known to be equivalent. But dependence of the equivalence constants on the order and the domain are more involved.

In this paper we analyze the equivalence of different variants of fractional-order semi-norms of positive orders bounded by one. The use of semi-norms is essential to guarantee scaling properties and we don't know of any publication that analyzes their equivalence. Instead of considering general smooth transformations that do not distort domains, we focus on pure isotropic scalings of domains. (More general transformations can be generated by composition with translations and area preserving smooth mappings.) In this way we keep appearing expressions of equivalence numbers as transparent as possible.

The rest of the paper is organized as follows. In Section 2 we collect all definitions and technical results. In Section 2.1 we recall two definitions of norms and define three different semi-norms: one of the Sobolev-Slobodeckij type, one by interpolation, and one of a quotient space type. Section 2.2 is devoted to basic equivalence estimates. In particular, we present Poincaré-Friedrichs' inequalities for the Sobolev-Slobodeckij and the interpolation semi-norm (Propositions 2.2 and 2.6). Their proofs are standard and presented here for completeness. Scaling properties of norms (also given for completeness) and semi-norms are analyzed in Section 2.3. Eventually, in Section 3 we combine the intermediate results to show the uniform (under scaling) equivalence of the Sobolev-Slobodeckij and the interpolation semi-norms (Theorem 3.1). Furthermore, we show that the Sobolev-Slobodeckij and quotient space semi-norms are uniformly equivalent (under scaling) as long as the scaling parameter is bounded from above (Theorem 3.2 with scaling bound 1 for simplicity). Theorem 3.3 shows that the interpolation and quotient space semi-norms can be uniformly bounded mutually in one direction depending on whether the scaling parameter is bounded from above or from below.

2 Sobolev norms

In this section we recall definitions of several Sobolev (semi-) norms and collect technical results that are needed to prove our main results in Section 3, or which are interesting in its own.

Throughout the paper, $\mathcal{O} \subset \mathbb{R}^n$ denotes a generic bounded connected Lipschitz domain. Some results require fewer conditions but we always implicitly assume that $\mathcal{O}$ is non-empty. We consider the usual $L^2(\mathcal{O})$- and $H^1(\mathcal{O})$-norms with notations $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{1,\mathcal{O}}$, respectively and the $H^1(\mathcal{O})$-semi-norm $|\cdot|_{1,\mathcal{O}}$. Here and in the following, in all types of norms, the underlying domain of definition $\mathcal{O}$ will be occasionally dropped from the notation when not being ambiguous.

2.1 Fractional-order norms and semi-norms

There are several ways to define Sobolev norms. We use the ones defined by a double integral (Sobolev-Slobodeckij) and by interpolation. For the latter we use the so-called real K-method, cf. [2]. For $0 < s < 1$, the interpolation norm in the fractional-order Sobolev space $H^s(\mathcal{O})$ is

defined by

$$\|v\|_{[L^2(\mathcal{O}),H^1(\mathcal{O})]_s} := \|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} := \left(\int_0^\infty t^{-2s} \inf_{v=v_0+v_1} \left(\|v_0\|_{0,\mathcal{O}}^2 + t^2 \|v_1\|_{1,\mathcal{O}}^2\right) \frac{dt}{t}\right)^{1/2}.$$

We also define the interpolation space

$$\tilde{H}^s(\mathcal{O}) = \left[L^2(\mathcal{O}), H_0^1(\mathcal{O})\right]_s$$

with corresponding notation for the norm. The notation $\tilde{H}^s$ is used by Grisvard and is common in the boundary element literature, whereas the notation $H_{00}^s = \tilde{H}^s$ is used by Lions and Magenes and is common in the finite element literature.

The Sobolev-Slobodeckij variant of these norms is defined (for $0 < s < 1$) by

$$\|v\|_{H^s(\mathcal{O})} := \|v\|_{s,\mathcal{O}} := \left(\|v\|_{L^2(\mathcal{O})}^2 + \int_\mathcal{O}\int_\mathcal{O} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}, \tag{2.1}$$

$$\|v\|_{\tilde{H}^s(\mathcal{O})} := \|v\|_{\sim,s,\mathcal{O}} := \left(\|v\|_{H^s(\mathcal{O})}^2 + \|\frac{v(x)}{\operatorname{dist}(x,\partial\mathcal{O})^s}\|_{L^2(\mathcal{O})}^2\right)^{1/2} \qquad \text{(preliminary version).}$$

The corresponding semi-norms are

$$|v|_{[L^2(\mathcal{O}),H^1(\mathcal{O})]_s} := |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} := \left(\int_0^\infty t^{-2s} \inf_{v=v_0+v_1} \left(\|v_0\|_{0,\mathcal{O}}^2 + t^2 |v_1|_{1,\mathcal{O}}^2\right) \frac{dt}{t}\right)^{1/2}$$

and

$$|v|_{H^s(\mathcal{O})} := |v|_{s,\mathcal{O}} := \left(\int_\mathcal{O}\int_\mathcal{O} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

Additionally, it is useful to define the semi-norm of quotient space type

$$|v|_{s,\mathcal{O},\inf} := \|v\|_{H^s(\mathcal{O})/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|v+c\|_{s,\mathcal{O}}.$$

2.2 Equivalence of semi-norms on a fixed domain

The aim of this section is to study equivalences of the semi-norms previously defined, on a fixed domain. Together with scaling properties (provided in Section 2.3) these estimates are needed to prove our main results in Section 3. Our proofs are based on a standard norm equivalence and specific Poincaré-Friedrichs' inequalities, which are also recalled here.

It is well known that different definitions of Sobolev norms are equivalent. However, equivalence constants depend usually on the order and the domain under consideration. In particular, for a bounded Lipschitz domain $\mathcal{O}$, the norms $\|\cdot\|_{s,\mathcal{O}}$ and $\|\cdot\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ are equivalent for $0 < s < 1$, cf. [9, 6, 10]. Such equivalences are shown by corresponding equivalences on $\mathbb{R}^n$ and the use of appropriate extension operators, cf. [3]. In particular, the norms previously defined are uniformly equivalent for s in a closed subset of $(0, 1)$, see [7]. Here, for the norms, we don't elaborate on the dependence of the equivalence constants on s and $\mathcal{O}$. We rather give them specific names to be used in estimates to follow.

Proposition 2.1 (equivalence of norms)**.** *For a fixed bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^n$ and for given $s \in (0,1)$ there exist constants $k(s,\mathcal{O}), K(s,\mathcal{O}) > 0$ such that*

$$k(s,\mathcal{O})\, \|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \le \|v\|_{s,\mathcal{O}} \le K(s,\mathcal{O})\, \|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \qquad \forall v \in H^s(\mathcal{O}).$$

For a proof see, e.g., [10].

It is well known that, on bounded domains, lower-order norms can be bounded by higher-order semi-norms plus finite rank terms. Such estimates are referred to as Poincaré-Friedrichs' inequalities. For integer-order norms there are direct proofs with explicit constants (depending on orders and domains) [12, Théorème 1.3] and attention has received finding best constants and deriving improved weighted estimates, see, e.g., [13, 14] and [4], respectively. We don't know of any direct proof for fractional-order norms on bounded domains (for unbounded domains however, see [11]). Therefore, for completeness, we state the result and give a proof (which is indirect and relies on the well-known compactness argument).

Proposition 2.2 (Poincaré-Friedrichs inequality, Sobolev-Slobodeckij semi-norm)**.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded connected domain, and $s \in (0,1)$. Then there exists a constant $C_{\mathrm{PF,SS}} > 0$, depending on $\mathcal{O}$ and s, such that*

$$\|v\|_{0,\mathcal{O}} \le C_{\mathrm{PF,SS}}(\mathcal{O},s)\Big(|v|_{s,\mathcal{O}} + |\int_{\mathcal{O}} v|\Big) \qquad \forall v \in H^s(\mathcal{O}).$$

Proof. Assume that the inequality is not true. Then there is a sequence $(v_j) \subset H^s(\mathcal{O})$ such that

$$\|v_j\|_{0,\mathcal{O}} = 1, \quad |v_j|_{s,\mathcal{O}} + |\int_{\mathcal{O}} v_j| \to 0 \ (j \to \infty).$$

Therefore, (v_j) is bounded in $H^s(\mathcal{O})$ and by Rellich's theorem (see [10, Theorem 3.27]) there is a convergent subsequence (again denoted by (v_j)) in $L^2(\mathcal{O})$. (Here, we use the Sobolev-Slobodeckij norm. But any equivalent norm would serve as well.) Since $|v_j|_{s,\mathcal{O}} \to 0$ this sequence is Cauchy and with limit v in $H^s(\mathcal{O})$. It holds $|v|_{s,\mathcal{O}} = 0$ so that v is constant. Furthermore, since $\int_{\mathcal{O}} v = 0$ and $\mathcal{O}$ is connected we conclude that $v = 0$, a contradiction to $\|v_j\|_{0,\mathcal{O}} = 1$. $\square$

Lemma 2.3. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded connected domain. Then there holds*

$$|v|^2_{s,\mathcal{O}} \le |v|^2_{s,\mathcal{O},\inf} = |v|^2_{s,\mathcal{O}} + \inf_{c\in\mathbb{R}} \|v+c\|^2_{0,\mathcal{O}} \le (1 + C^2_{\mathrm{PF,SS}})|v|^2_{s,\mathcal{O}}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $C_{\mathrm{PF,SS}} = C_{\mathrm{PF,SS}}(\mathcal{O},s)$ is the number from Proposition 2.2.

Proof. By definition of $|\cdot|_{s,\mathcal{O}}$ there holds for any $c \in \mathbb{R}$ and any $v \in H^s(\mathcal{O})$ (we now drop $\mathcal{O}$ from the notation)

$$|v|_s = |v+c|_s.$$

Therefore

$$|v|_s \le \inf_{c\in\mathbb{R}} \|v+c\|_s = |v|_{s,\inf}$$

which is the first assertion. By the initial argument and the definition of the Sobolev-Slobodeckij norm one also finds that

$$|v|_{s,\inf}^2 = \inf_{c\in\mathbb{R}} \|v+c\|_s^2 = \inf_{c\in\mathbb{R}} \|v+c\|_0^2 + |v|_s^2.$$

This is the second assertion.

The last relation and the Poincaré-Friedrichs' inequality (Proposition 2.2) lead to

$$|v|_{s,\inf}^2 \le C_{\mathrm{PF,SS}}^2 \inf_{c\in\mathbb{R}} \Big(|v|_s + |\int_{\mathcal{O}} (v+c)|\Big)^2 + |v|_s^2 = (1+C_{\mathrm{PF,SS}}^2)|v|_s^2.$$

This finishes the proof. ☐

Lemma 2.4. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There holds*

$$k^2|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \le |v|_{s,\mathcal{O},\inf}^2 \le 3K^2|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 + \frac{K^2}{s(1-s)} \inf_{c\in\mathbb{R}} \|v+c\|_{0,\mathcal{O}}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $k = k(s,\mathcal{O})$ and $K = K(s,\mathcal{O})$ are the numbers from Proposition 2.1.

Proof. Let $v \in H^s(\mathcal{O})$, and let c_0, c_1 denote generic constants. For any $t > 0$ there holds

$$\begin{aligned}\inf_{v=v_0+v_1} \Big(\|v_0\|_0^2 + t^2|v_1|_1^2\Big) &= \inf_{v=v_0+c_0+v_1+c_1} \Big(\|v_0+c_0\|_0^2 + t^2|v_1|_1^2\Big)\\ &= \inf_{c_1, v-c_1=v_0+v_1} \Big(\|v_0\|_0^2 + t^2|v_1|_1^2\Big),\end{aligned}$$

that is

$$\begin{aligned}\inf_{v=v_0+v_1} \Big(\|v_0\|_0^2 + t^2|v_1|_1^2\Big) &= \inf_{c\in\mathbb{R}} \inf_{v+c=v_0+v_1} \Big(\|v_0\|_0^2 + t^2|v_1|_1^2\Big)\\ &\le \inf_{c\in\mathbb{R}} \inf_{v+c=v_0+v_1} \Big(\|v_0\|_0^2 + t^2\|v_1\|_1^2\Big).\end{aligned}$$

We conclude that

$$\begin{aligned}|v|_{L^2,H^1,s}^2 &= \int_0^\infty t^{-2s} \inf_{v=v_0+v_1} \Big(\|v_0\|_0^2 + t^2\,|v_1|_1^2\Big)\frac{dt}{t}\\ &\le \inf_{c\in\mathbb{R}} \int_0^\infty t^{-2s} \inf_{v+c=v_0+v_1} \Big(\|v_0\|_0^2 + t^2\,\|v_1\|_1^2\Big)\frac{dt}{t} = \inf_{c\in\mathbb{R}} \|v+c\|_{L^2,H^1,s}^2.\end{aligned}$$

By Proposition 2.1

$$\inf_{c\in\mathbb{R}} \|v+c\|_{L^2,H^1,s}^2 \le k^{-2} \inf_{c\in\mathbb{R}} \|v+c\|_s^2 = k^{-2}|v|_{s,\inf}^2,$$

so that the first assertion follows.

By definition and using Proposition 2.1 there holds

$$\begin{aligned} |v|^2_{s,\inf} &= \inf_{c\in\mathbb{R}} \|v+c\|^2_s \le K^2 \inf_{c\in\mathbb{R}} \|v+c\|^2_{L^2,H^1,s} \\ &= K^2 \inf_{c\in\mathbb{R}} \int_0^\infty t^{-2s} \inf_{v+c=v_0+v_1} \Big(\|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2\Big)\frac{dt}{t}. \end{aligned} \tag{2.2}$$

We bound the integrand separately for $t<1$ and $t\ge 1$.

For $t<1$ we use the representation $v+c=v_0+v_1$ to bound

$$\begin{aligned} \|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2 &\le \|v_0\|_0^2 + 2t^2\big(\|v+c\|_0^2 + \|v_0\|_0^2\big) + t^2|v_1|_1^2 \\ &\le 3\|v_0\|_0^2 + 2t^2\|v+c\|_0^2 + t^2|v_1|_1^2. \end{aligned}$$

If $t\ge 1$ then we select $v_0 := v+c$ to conclude that

$$\inf_{v+c=v_0+v_1}\Big(\|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2\Big) \le \|v+c\|_0^2.$$

Together this yields

$$\begin{aligned} &\int_0^\infty t^{-2s} \inf_{v+c=v_0+v_1}\Big(\|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2\Big)\frac{dt}{t} \\ &\le \int_0^1 t^{-2s} \inf_{v+c=v_0+v_1}\Big(3\|v_0\|_0^2 + 2t^2\|v+c\|_0^2 + t^2|v_1|_1^2\Big)\frac{dt}{t} + \int_1^\infty t^{-2s}\|v+c\|_0^2\frac{dt}{t} \\ &= \int_0^1 t^{-2s} \inf_{v+c=v_0+v_1}\Big(3\|v_0\|_0^2 + t^2|v_1|_1^2\Big)\frac{dt}{t} + \|v+c\|_0^2\Big(\int_0^1 2t^{1-2s}\,dt + \int_1^\infty t^{-1-2s}\,dt\Big) \\ &\le 3|v|^2_{L^2,H^1,s} + \frac{1}{s(1-s)}\|v+c\|_0^2. \end{aligned} \tag{2.3}$$

Therefore, recalling (2.2), we obtain

$$|v|^2_{s,\inf} \le 3K^2|v|^2_{L^2,H^1,s} + \frac{K^2}{s(1-s)}\inf_{c\in\mathbb{R}}\|v+c\|_0^2,$$

which is the second assertion. □

From the proof of the previous lemma one can conclude that the semi-norm $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ is indeed the principal part of a norm in $H^s(\mathcal{O})$. This will be useful to deduce a Poincaré-Friedrichs inequality with this semi-norm. First let us specify what we mean by the semi-norm being principal part of a norm.

Corollary 2.5. *Let $\mathcal{O}\subset\mathbb{R}^n$ be a connected bounded Lipschitz domain. There holds*

$$\|v\|^2_{s,\mathcal{O}} \le \frac{K^2}{s(1-s)}\|v\|^2_{0,\mathcal{O}} + 3K^2|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$

for any $v\in H^s(\mathcal{O})$ and $s\in(0,1)$. Here, $K=K(s,\mathcal{O})$ is the number from Proposition 2.1.

Proof. This is a combination of the second bound from Proposition 2.1 and (2.3) with $c = 0$. $\square$

We are now ready to prove a second Poincaré-Friedrichs inequality.

Proposition 2.6 (Poincaré-Friedrichs inequality, interpolation semi-norm)**.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, and $s \in (0,1)$. Then there exists a constant $C_{\mathrm{PF,I}} > 0$, depending on $\mathcal{O}$ and s, such that*

$$\|v\|_{0,\mathcal{O}} \le C_{\mathrm{PF,I}}(\mathcal{O}, s)\Big(|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} + |\int_{\mathcal{O}} v|\Big) \qquad \forall v \in H^s(\mathcal{O}).$$

Proof. The proof is identical to the one of Proposition 2.2 by noting the following two facts. First, due to Corollary 2.5, $\|v_j\|_{0,\mathcal{O}} = 1$ and $|v_j|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \to 0$ imply that (v_j) is bounded in $H^s(\mathcal{O})$. Second, $|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} = 0$ implies that v is constant. $\square$

With the help of Proposition 2.6 we can now turn the estimate by Lemma 2.4 into a semi-norm equivalence.

Lemma 2.7. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a connected bounded Lipschitz domain. There holds*

$$k^2 |v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \le |v|^2_{s,\mathcal{O},\inf} \le K^2\Big(3 + \frac{C^2_{\mathrm{PF,I}}}{s(1-s)}\Big)|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $k = k(s,\mathcal{O})$, $K = K(s,\mathcal{O})$ are the numbers from Proposition 2.1, and $C_{\mathrm{PF,I}} = C_{\mathrm{PF,I}}(s,\mathcal{O})$ is the number from Proposition 2.6.

Proof. The lower bound is the one from Lemma 2.4. The upper bound is a combination of the one from the same lemma and the Poincaré-Friedrichs' inequality from Proposition 2.6. $\square$

Meanwhile we have accumulated quite some parameters in the semi-norm estimates that depend on the order s and the domain $\mathcal{O}$ under consideration. Our goal is to show equivalence of semi-norms which is uniform for a family of scaled domains. We therefore study scaling properties of semi-norms in the following section. In this way, parameters from this section enter final results only via their values on a reference domain.

2.3 Scalability of norms and semi-norms

Obviously, both norms in $H^s(\mathcal{O})$ defined previously, $\|\cdot\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),h}$ and $\|\cdot\|_{s,\mathcal{O}}$, are not scalable. This could be achieved by weighting the $L^2(\mathcal{O})$-contributions according to the diameter of $\mathcal{O}$, for instance, cf. [5]. Of course, in this way one does not obtain uniformly equivalent norms (of un-weighted and weighted variants) under transformation of the domain.

This is different for the norm in $\tilde{H}^s(\mathcal{O})$. It can be easily fixed (to be scalable) by using that the semi-norm $|\cdot|_{1,\mathcal{O}}$ is a norm in $H^1_0(\mathcal{O})$, and re-defining

$$\|v\|_{[L^2(\mathcal{O}),H^1_0(\mathcal{O})]_s} := \|v\|_{L^2(\mathcal{O}),H^1_0(\mathcal{O}),s} := \Big(\int_0^\infty t^{-2s} \inf_{v=v_0+v_1, v_1\in H^1_0(\mathcal{O})} \Big(\|v_0\|^2_{0,\mathcal{O}} + t^2\, |v_1|^2_{1,\mathcal{O}}\Big)\frac{dt}{t}\Big)^{1/2}$$

in the case of interpolation. In the case of the Sobolev-Slobodeckij norm one can ensure scalability by re-defining

$$\|v\|_{\tilde{H}^s(\mathcal{O})} := \|v\|_{\sim,s,\mathcal{O}} := \left(|v|^2_{H^s(\mathcal{O})} + \|\frac{v(x)}{\operatorname{dist}(x,\partial\mathcal{O})^s}\|^2_{L^2(\mathcal{O})} \right)^{1/2}$$

since the last term guarantees positivity. In the following we will make use of these re-defined norms.

For a domain $\mathcal{O} \in \mathbb{R}^n$ we denote by $\mathcal{O}_h$ the scaled domain

$$\mathcal{O}_h := \{D_h x;\, x \in \mathcal{O}\} \quad \text{with} \quad D_h := \operatorname{diag}(h,\dots,h) \in \mathbb{R}^{n\times n}.$$

Correspondingly, for a given real function v defined on $\mathcal{O}$,

$$v_h : \begin{cases} \mathcal{O}_h & \to \quad \mathbb{R} \\ x_h = D_h x & \mapsto \quad v(x) \end{cases}$$

is the function transformed onto $\mathcal{O}_h$ by scaling.

Lemma 2.8 (scalability of norms)**.** *For a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^n$, $s \in (0,1)$ and $v \in \tilde{H}^s(\mathcal{O})$ there hold the scaling properties*

$$\|v_h\|^2_{L^2(\mathcal{O}_h),H^1_0(\mathcal{O}_h),s} = h^{n-2s}\|v\|^2_{L^2(\mathcal{O}),H^1_0(\mathcal{O}),s},$$
$$\|v_h\|^2_{\sim,s,\mathcal{O}_h} = h^{n-2s}\|v\|^2_{\sim,s,\mathcal{O}}.$$

Proof. For the interpolation norm and $\mathcal{O}$ being a cube, this property (with an unspecified equivalence constant) has been shown in [7]. It is simply the scaling properties of the L^2 and H^1_0-norms together with the exactness of the K-method of interpolation (employed here). However, the scaling (with constant 1) is immediate from the definition with transformation $t = hr$ and using the scaling properties of the L^2 and H^1_0-norms:

$$\begin{aligned}
\|v_h\|^2_{L^2(\mathcal{O}_h),H^1_0(\mathcal{O}_h),s} &= \int_0^\infty t^{-2s} \inf_{v_h = v_{0,h}+v_{1,h},\, v_{1,h}\in H^1_0(\mathcal{O}_h)} \Big(\|v_{0,h}\|^2_{0,\mathcal{O}_h} + t^2\, |v_{1,h}|^2_{1,\mathcal{O}_h} \Big) \frac{dt}{t} \\
&= \int_0^\infty t^{-2s} \inf_{v = v_0+v_1,\, v_1\in H^1_0(\mathcal{O})} \Big(h^n \|v_0\|^2_{0,\mathcal{O}} + t^2\, h^{n-2} |v_1|^2_{1,\mathcal{O}} \Big) \frac{dt}{t} \\
&= h^n \int_0^\infty (hr)^{-2s} \inf_{v = v_0+v_1,\, v_1\in H^1_0(\mathcal{O})} \Big(\|v_0\|^2_{0,\mathcal{O}} + r^2\, |v_1|^2_{1,\mathcal{O}} \Big) \frac{dr}{r} \\
&= h^{n-2s} \|v\|^2_{L^2(\mathcal{O}),H^1_0(\mathcal{O}),s}.
\end{aligned}$$

This proves the first assertion. The scaling property of the second norm is obtained also by transformation ($x_h = D_h x$, $y_h = D_h y$):

$$\begin{aligned}
\|v_h\|^2_{\sim,s,\mathcal{O}_h} &= \int_{\mathcal{O}_h}\int_{\mathcal{O}_h} \frac{|v_h(x_h) - v_h(y_h)|^2}{|x_h - y_h|^{n+2s}}\, dx_h\, dy_h + \int_{\mathcal{O}_h} \Big(\frac{v_h(x_h)}{\operatorname{dist}(x_h,\partial\mathcal{O}_h)^s} \Big)^2 dx_h \\
&= h^{2n} \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{h^{n+2s}|x-y|^{n+2s}}\, dx\, dy + h^n \int_{\mathcal{O}_h} \Big(\frac{v(x)}{h^s \operatorname{dist}(x,\partial\mathcal{O})^s} \Big)^2 dx \\
&= h^{n-2s} \|v\|^2_{\sim,s,\mathcal{O}}.
\end{aligned}$$

$\square$

Lemma 2.9 (scalability of semi-norms)**.** *For a bounded domain $\mathcal{O} \subset \mathbb{R}^n$, $s \in (0,1)$ and $v \in H^s(\mathcal{O})$ there hold the scaling properties*

$$|v_h|^2_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s} = h^{n-2s}|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s},$$
$$|v_h|^2_{s,\mathcal{O}_h} = h^{n-2s}|v|^2_{s,\mathcal{O}}.$$

Proof. The proof is basically identical to the one of Lemma 2.8. $\square$

The third semi-norm, $|\cdot|_{s,\mathcal{O},\mathrm{inf}}$, is obviously not scalable. Instead, there holds the following.

Lemma 2.10. *For a bounded domain $\mathcal{O} \subset \mathbb{R}^n$, $s \in (0,1)$ and $v \in H^s(\mathcal{O})$ there holds*

$$|v_h|^2_{s,\mathcal{O}_h,\mathrm{inf}} = h^{n-2s}|v|^2_{s,\mathcal{O}} + h^n \inf_{c\in\mathbb{R}} \|v+c\|^2_{0,\mathcal{O}}.$$

Proof. This result is immediate from the representation of the semi-norm given in Lemma 2.3 and the scaling properties of the $|\cdot|_s$-semi-norm by Lemma 2.9 and of the L^2-norm. $\square$

3 Main results

We are now ready to state and prove our main results on certain equivalences of fractional-order Sobolev semi-norms. We use the notation for scaling from Section 2.3.

The first theorem shows for fixed $s \in (0,1)$ the uniform equivalence of the semi-norms $|\cdot|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s}$ and $|\cdot|_{s,\mathcal{O}_h}$ on a family of scaled domains $\mathcal{O}_h$.

Theorem 3.1. *Let $\mathcal{O}_h \subset \mathbb{R}^n$ be a family of domains that is obtained by scaling with $h > 0$ of a fixed bounded connected Lipschitz domain $\mathcal{O}$. Then there hold the following relations.*

(i)
$$|v|_{s,\mathcal{O}_h} \le C_1(s)|v|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s} \qquad \forall v \in H^s(\mathcal{O}_h),\ \forall s \in (0,1),\ \forall h > 0.$$
Here, $C_1(s) > 0$ depends on s but is independent of h and v:
$$C_1^2 = C_1(s,\mathcal{O})^2 = K(s,\mathcal{O})^2\Big(3 + \frac{C_{\mathrm{PF,I}}(s,\mathcal{O})^2}{s(1-s)}\Big)$$
with $K(s,\mathcal{O})$ from Proposition 2.1 and $C_{\mathrm{PF,I}}(s,\mathcal{O})$ from Proposition 2.6.

(ii)
$$|v|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s} \le C_2(s)|v|_{s,\mathcal{O}_h} \qquad \forall v \in H^s(\mathcal{O}_h),\ \forall s \in (0,1),\ \forall h > 0.$$
Here, $C_2(s) > 0$ depends on s but is independent of h and v:
$$C_2^2 = C_2(s,\mathcal{O})^2 = k(s,\mathcal{O})^{-2}\Big(1 + C_{\mathrm{PF,SS}}(s,\mathcal{O})^2\Big)$$
with $k(s,\mathcal{O})$ from Proposition 2.1 and $C_{\mathrm{PF,SS}}(s,\mathcal{O})$ from Proposition 2.2.

Proof. On a fixed domain $\mathcal{O}$ we obtain, by combining Lemmas 2.3 and 2.7, the equivalence of semi-norms:

$$|v|^2_{s,\mathcal{O}} \le |v|^2_{s,\mathcal{O},\inf} \le K(s,\mathcal{O})^2\Big(3 + \frac{C_{\mathrm{PF,I}}(s,\mathcal{O})^2}{s(1-s)}\Big)|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$

and

$$|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \le k(s,\mathcal{O})^{-2}|v|^2_{s,\mathcal{O},\inf} \le k(s,\mathcal{O})^{-2}\Big(1 + C_{\mathrm{PF,SS}}(s,\mathcal{O})^2\Big)|v|^2_{s,\mathcal{O}}$$

Both assertions of the theorem follow by the scaling properties of the semi-norms, see Lemma 2.9. □

The next two theorems study the other pairs of semi-norms for equivalence, $(|\cdot|_{s,\mathcal{O}_h}, |\cdot|_{s,\mathcal{O}_h,\inf})$ and $(|\cdot|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s}, |\cdot|_{s,\mathcal{O}_h,\inf})$.

Theorem 3.2. *Let $\mathcal{O}_h \subset \mathbb{R}^n$ be a family of domains that is obtained by scaling with $h > 0$ of a fixed bounded connected Lipschitz domain $\mathcal{O}$. Then there hold the following relations.*

(i)

$$|v|_{s,\mathcal{O}_h} \le |v|_{s,\mathcal{O}_h,\inf} \qquad \forall v \in H^s(\mathcal{O}_h),\ \forall s \in (0,1),\ \forall h > 0.$$

(ii)

$$|v|_{s,\mathcal{O}_h,\inf} \le C(s)|v|_{s,\mathcal{O}_h} \qquad \forall v \in H^s(\mathcal{O}_h),\ \forall s \in (0,1),\ \forall h \in (0,1].$$

Here, $C(s)^2 = 1 + C_{\mathrm{PF,SS}}(s,\mathcal{O})^2$ is independent of h and v, with $C_{\mathrm{PF,SS}}$ being the number from Proposition 2.2.

Proof. Assertion (i) is a repetition of the first estimate in Lemma 2.3. The second assertion is a combination of the second estimate in Lemma 2.3 with scaling properties provided by Lemmas 2.10 and 2.9:

$$\begin{aligned}|v_h|^2_{s,\mathcal{O}_h,\inf} &= h^{n-2s}\Big(h^{2s}\inf_{c\in\mathbb{R}}\|v+c\|^2_{0,\mathcal{O}} + |v|^2_{s,\mathcal{O}}\Big)\\ &\le h^{n-2s}\Big(1 + C_{\mathrm{PF,SS}}(s,\mathcal{O})^2\Big)|v|^2_{s,\mathcal{O}} = \Big(1 + C_{\mathrm{PF,SS}}(s,\mathcal{O})^2\Big)|v_h|^2_{s,\mathcal{O}_h}.\end{aligned}$$

Here we used the notation v_h of a scaled function from Section 2.3, and the condition that $h \le 1$. □

Theorem 3.3. *Let $\mathcal{O}_h \subset \mathbb{R}^n$ be a family of domains that is obtained by scaling with $h > 0$ of a fixed bounded connected Lipschitz domain $\mathcal{O}$. Then there hold the following relations.*

(i)

$$|v|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s} \le C_1(s)|v|_{s,\mathcal{O}_h,\inf} \qquad \forall v \in H^s(\mathcal{O}_h),\ \forall s \in (0,1),\ \forall h \ge 1.$$

Here, $C_1(s) > 0$ depends on s but is independent of h and v:

$$C_1(s)^2 = k(s,\mathcal{O})^{-2}$$

with $k(s,\mathcal{O})$ from Proposition 2.1.

(ii)

$$|v|_{s,\mathcal{O}_h,\inf} \le C_2(s)|v|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s} \qquad \forall v \in H^s(\mathcal{O}_h),\ \forall s \in (0,1),\ \forall h \in (0,1].$$

Here, $C_2(s) > 0$ depends on s but is independent of h and v:

$$C_2(s)^2 = K(s,\mathcal{O})^2\Big(3 + \frac{C_{\mathrm{PF,I}}(s,\mathcal{O})^2}{s(1-s)}\Big)$$

with $K(s,\mathcal{O})$ from Proposition 2.1 and $C_{\mathrm{PF,I}}(s,\mathcal{O})$ from Proposition 2.6.

Proof. We use the notation for scaled functions v_h from Section 2.3. By scaling properties provided by Lemma 2.9 and the first estimate of Lemma 2.7, together with the restriction $h \ge 1$, we obtain

$$\begin{aligned}|v_h|^2_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s} &= h^{n-2s}|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \le k(s,\mathcal{O})^{-2}h^{n-2s}|v|^2_{s,\mathcal{O},\inf} \\ &= k(s,\mathcal{O})^{-2}\Big(|v_h|^2_{s,\mathcal{O}_h} + h^{-2s}\inf_{c\in\mathbb{R}}\|v_h+c\|^2_{0,\mathcal{O}_h}\Big) \le k(s,\mathcal{O})^{-2}|v_h|^2_{s,\mathcal{O}_h,\inf}.\end{aligned}$$

This is the first assertion. To show the second one we use the scaling properties in Lemmas 2.10, 2.9, the condition $h \le 1$, and Lemma 2.7:

$$\begin{aligned}|v_h|^2_{s,\mathcal{O}_h,\inf} &= h^{n-2s}|v|^2_{s,\mathcal{O}} + h^n\inf_{c\in\mathbb{R}}\|v+c\|^2_{0,\mathcal{O}} \le h^{n-2s}|v|^2_{s,\mathcal{O},\inf} \\ &\le h^{n-2s}K(s,\mathcal{O})^2\Big(3+\frac{C_{\mathrm{PF,I}}(s,\mathcal{O})^2}{s(1-s)}\Big)|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \\ &= K(s,\mathcal{O})^2\Big(3+\frac{C_{\mathrm{PF,I}}(s,\mathcal{O})^2}{s(1-s)}\Big)|v_h|^2_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s}.\end{aligned}$$

□

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